

Kinematics of a Particle

In this chapter, we analyze the motion of bodies without investigating its causes.

The Particle

To facilitate our observations of an object's motion, we can describe the state of a body using spatial coordinates relative to a point. We call this point **material point** or **particle**. It is an idealized body in which all the mass is concentrated at a single point.

The Position Vector

To describe the position of a body at time t we use a coordinate system. In most cases, we will use a three-dimensional orthonormal coordinate system. The three coordinates x, y, z therefore describe the position of the observed material point m (or, in the case of a rigid body, its center of mass). Along a certain path Γ , the three coordinates will be functions of time:

$$x(t), y(t), z(t) \tag{1}$$

We can now define the **position vector** $\mathbf{r}(t)$:

Position Vector

$$\mathbf{r}(t) := \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} \tag{2}$$

The position vector is defined with respect to the origin of the reference frame¹ and therefore cannot be freely translated.

To calculate the **length** of a position vector, we use the following formula:

$$r := |\mathbf{r}| = \sqrt{x^2 + y^2 + z^2} \tag{3}$$

For the **direction** of a position vector, we use the symbol $\hat{\mathbf{r}}$:

$$\mathbf{e}_r \equiv \hat{\mathbf{r}} := \frac{\mathbf{r}}{r} \tag{4}$$

This vector always has a length of 1.

The **displacement vector** gives the distance between the position at time t_1 and time t_2 , that is, both the distance and the direction of the displacement of the material point (see Figure 1.1):

¹The coordinate system chosen for a physical problem is generally called a reference frame.

$$\Delta \mathbf{r} = \mathbf{r}(t_2) - \mathbf{r}(t_1) \quad (5)$$

Conversely, the distance traveled along the curve Γ is given by:

$$\int_{\Gamma} |d\mathbf{r}| = \int_{\Gamma} \sqrt{(dx)^2 + (dy)^2 + (dz)^2} \quad (6)$$

Figure 1: Vector representation of motion: position and displacement vectors.

Velocity

Velocity as a vector

Velocity \mathbf{v} , like position, is also a vector; it indicates not only how fast we are moving but also in which direction. To calculate the **average velocity** $\langle \mathbf{v} \rangle$, divide the displacement vector by the respective time interval:

Average Velocity

$$\langle \mathbf{v} \rangle := \frac{\Delta \mathbf{r}}{\Delta t} \quad (7)$$

Considering the limit as $\Delta t \rightarrow 0$, we obtain the instantaneous velocity:

Instantaneous Velocity

$$\mathbf{v}(t) := \lim_{\Delta t \rightarrow 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t} = \frac{d\mathbf{r}(t)}{dt} = \dot{\mathbf{r}}(t) \quad (8)$$

To obtain the three components of the velocity vector, differentiate the three components of the position vector:

$$\mathbf{v}(t) = \dot{\mathbf{r}}(t) = \begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \\ \dot{z}(t) \end{pmatrix} \quad (9)$$

Velocity is always tangent to the path.

Uniform Rectilinear Motion

As a first example, consider a body moving with constant velocity v_0 along a straight line g passing through the origin of our reference frame. We choose the velocity such that it lies in the xz plane and is inclined at a certain angle α with respect to the x -axis. The three velocity components are:

$$\mathbf{v}(t) = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = \begin{pmatrix} v_0 \cos(\alpha) \\ 0 \\ v_0 \sin(\alpha) \end{pmatrix} \quad (10)$$

Integrating the velocity components, we obtain the position vector:

$$\mathbf{r}(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} x_0 + v_0 t \cos(\alpha) \\ y_0 \\ z_0 + v_0 t \sin(\alpha) \end{pmatrix} \quad (11)$$

x_0 , y_0 and z_0 are the constants of integration and represent the initial position of the body at time $t_0 = 0$. We can choose the reference frame such that the body is located at the origin at time t_0 , to simplify calculations.

Acceleration

Acceleration as a vector

The **acceleration** \mathbf{a} is also a vector. Indeed, we are interested not only in how much we are accelerating, but also in which direction.

We define the **average acceleration** as the ratio of the change in velocity over a certain time interval:

Average Acceleration

$$\langle \mathbf{a} \rangle := \frac{\Delta \mathbf{v}}{\Delta t} \quad (12)$$

Taking the limit as $\Delta t \rightarrow 0$, we obtain the **instantaneous acceleration**:

Instantaneous Acceleration

$$\mathbf{a}(t) := \lim_{\Delta t \rightarrow 0} \frac{\mathbf{v}(t + \Delta t) - \mathbf{v}(t)}{\Delta t} = \frac{d\mathbf{v}(t)}{dt} = \dot{\mathbf{v}}(t) \quad (13)$$

Acceleration is therefore the derivative of velocity and the second derivative of position:

$$\mathbf{a}(t) = \frac{d\mathbf{v}(t)}{dt} = \dot{\mathbf{v}}(t) = \frac{d^2\mathbf{r}(t)}{dt^2} = \ddot{\mathbf{r}}(t) \quad (14)$$

Or written in vector form with its components:

$$\mathbf{a}(t) = \begin{pmatrix} \dot{v}_x(t) \\ \dot{v}_y(t) \\ \dot{v}_z(t) \end{pmatrix} = \begin{pmatrix} \ddot{x}(t) \\ \ddot{y}(t) \\ \ddot{z}(t) \end{pmatrix} \quad (15)$$

Acceleration can cause a change in both the magnitude and the direction of the velocity. This means that a motion in which the magnitude of the velocity is constant could still be accelerated if the direction of the velocity were not constant (example: circular motion).

Constant Acceleration (Uniformly Accelerated Motion)

In this case, we consider a motion in which both the magnitude and the direction of the acceleration are constant.

$$\mathbf{a}(t) = \text{const.} \quad (16)$$

$$\implies \mathbf{v}(t) = \mathbf{v}(t_0) + \mathbf{a}_0 \int_{t_0}^t dt' \quad (17)$$

$$\implies \mathbf{v}(t) = \mathbf{v}_0 + \mathbf{a}_0 \Delta t \quad (18)$$

Similarly, we have:

$$\mathbf{r}(t) = \mathbf{r}_0 + \mathbf{v}_0 \Delta t + \frac{1}{2} \mathbf{a}_0 (\Delta t)^2 \quad (19)$$

Example 1. Free fall

Consider an object in free fall and choose a reference frame in which the z -axis points upwards. Then the acceleration is:

$$\mathbf{a}(t) = \begin{pmatrix} 0 \\ 0 \\ -g \end{pmatrix} \quad (20)$$

where g is the gravitational acceleration at the Earth's surface and is equal to 9.81 ms^{-2} .

Assuming the object has no initial velocity, we obtain the equation of motion for the body:

$$\implies \mathbf{r}(t) = \begin{pmatrix} x_0 \\ y_0 \\ z_0 - \frac{1}{2}gt^2 \end{pmatrix} \quad (21)$$

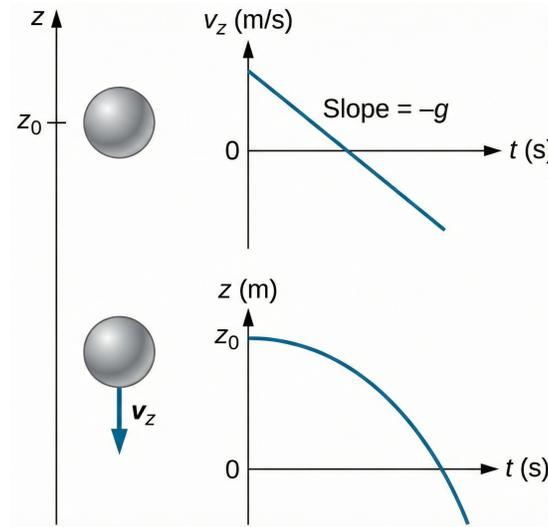


Figure 2: Free Fall Motion

Example 2. Parabolic motion

An object that is launched always follows a perfectly parabolic path (assuming it is not influenced by external factors). In this case, we can consider the components separately. As before, $\mathbf{a}_0 = \mathbf{g}$. This time, we consider as initial conditions:

$$\mathbf{r}(0) = \begin{pmatrix} 0 \\ 0 \\ z_0 \end{pmatrix}; \quad \mathbf{v}(0) = \begin{pmatrix} v_0 \cos(\alpha) \\ 0 \\ v_0 \sin(\alpha) \end{pmatrix} \quad (22)$$

Since the acceleration is directed exclusively downward, by decomposing the motion into its various components, we observe that horizontally the body moves with uniform rectilinear motion, while vertically it moves with uniformly accelerated motion:

$$\Rightarrow \mathbf{r}(t) = \begin{pmatrix} v_0 t \cos(\alpha) \\ 0 \\ z_0 + v_0 t - \frac{1}{2} g t^2 \end{pmatrix} \quad (23)$$

In this case, t acts as a parameter. We can eliminate t to obtain the equation of the parabolic trajectory $z(x)$:

$$t = \frac{x}{v_0 \cos(\alpha)} \quad (24)$$

$$z(x) = z_0 + \frac{x \sin(\alpha)}{\cos(\alpha)} - \frac{g}{2} \frac{x^2}{v_0^2 \cos^2(\alpha)} \quad (25)$$

Circular Motion

Consider a particle moving uniformly on a circle of radius r . We choose a reference frame whose origin coincides with the center of this circle, and such that the motion takes place in the xy plane. Using polar coordinates, we therefore have:

$$\rho(t) = r \quad (26)$$

The angle φ varies with time and is given by:

$$\varphi(t) = \varphi_0 + \omega t \quad (27)$$

where φ_0 indicates the angle at time $t = 0$. ω is the angular velocity, and in our case it is constant.

Now expressing the position vector in Cartesian coordinates, we obtain:

$$\mathbf{r}(t) = r \begin{pmatrix} \cos(\omega t + \varphi_0) \\ \sin(\omega t + \varphi_0) \\ 0 \end{pmatrix} \quad (28)$$

Consequently, the velocity is given by:

$$\mathbf{v}(t) = \dot{\mathbf{r}}(t) = r\omega \begin{pmatrix} -\sin(\omega t + \varphi_0) \\ \cos(\omega t + \varphi_0) \\ 0 \end{pmatrix} \quad (29)$$

The velocity vector is not constant. Its direction changes continuously, and is always perpendicular to the position vector.

In contrast, the magnitude of the velocity

$$|\mathbf{v}(t)| = v(t) = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} = r\omega \quad (30)$$

is constant.

We obtain acceleration differentiating once again:

$$\mathbf{a}(t) = \dot{\mathbf{v}}(t) = -r\omega^2 \begin{pmatrix} \cos(\omega t + \varphi_0) \\ \sin(\omega t + \varphi_0) \\ 0 \end{pmatrix} = -\omega^2 \mathbf{r}(t) \quad (31)$$

We notice that the acceleration vector has the same line of action as the position vector but points in the opposite direction. This acceleration, always directed towards the center, is called **centripetal acceleration**. We also note that in circular motion with constant angular velocity, the magnitude of \mathbf{a} is constant over time. Despite this, the direction changes at every instant.

Magnitude of centripetal acceleration The magnitude of the centripetal acceleration depends on the distance from the center of rotation r and is proportional to the square of the angular velocity:

$$a = r\omega^2 = \frac{v^2}{r} \quad (32)$$

Period and Frequency Through the constant angular velocity, we can calculate the period of rotation of a body:

$$T := \frac{2\pi}{\omega} = \frac{2\pi r}{v} \quad (33)$$

and the frequency of rotation:

$$\nu := \frac{1}{T} = \frac{\omega}{2\pi} = \frac{v}{2\pi r} \quad (34)$$

The relationship between angular and linear velocity is therefore:

$$\omega = \frac{v}{r} \quad (35)$$

Angular velocity as a vector

Consider now a circular motion taking place on a plane parallel to the xy plane at a distance z . The magnitude of the velocity with respect to the center of the circle is:

$$v = \rho\omega \quad (36)$$

We now define the **angular velocity vector** $\boldsymbol{\omega}$, perpendicular to the plane of circular motion, obtaining the following equation:

$$\mathbf{v} = \boldsymbol{\omega} \times \boldsymbol{\rho} \quad (37)$$

Angular Velocity Vector Using a well-known vector identity we get:

$$\boldsymbol{\omega} = \frac{1}{\rho^2}(\boldsymbol{\rho} \times \mathbf{v}) \quad (38)$$

Angular Acceleration

In the case where the angular velocity is not constant, we define the angular acceleration as

$$\alpha := \dot{\omega} = \ddot{\varphi} \quad (39)$$

We now rewrite the position vector in the plane of rotation, assuming $\varphi_0 = 0$:

$$\mathbf{r}(t) = r \begin{pmatrix} \cos\varphi(t) \\ \sin\varphi(t) \end{pmatrix} \quad (40)$$

Velocity is then equal to:

$$\mathbf{v}(t) = \dot{\mathbf{r}}(t) = r\dot{\varphi} \begin{pmatrix} -\sin\varphi(t) \\ \cos\varphi(t) \end{pmatrix} = r\omega \begin{pmatrix} -\sin\varphi(t) \\ \cos\varphi(t) \end{pmatrix} \quad (41)$$

The magnitude of velocity is given by:

$$v = r\omega(t) \quad (42)$$

which is therefore no longer constant in time.

Linear Acceleration Linear acceleration is given by differentiating velocity in respect to time:

$$\mathbf{a}(t) = \dot{\mathbf{v}}(t) = \underbrace{r\ddot{\varphi} \begin{pmatrix} -\sin\varphi(t) \\ \cos\varphi(t) \end{pmatrix}}_{=a_{\parallel}} - \underbrace{r\dot{\varphi}^2 \begin{pmatrix} \cos\varphi(t) \\ \sin\varphi(t) \end{pmatrix}}_{=a_{\perp}} \quad (43)$$

The acceleration is therefore expressed through two components. The first component is the tangential acceleration (a_{\parallel}), which is parallel to the linear velocity. The second component is the centripetal acceleration (a_{\perp}). The magnitudes of the two components are:

$$|\mathbf{a}_{\parallel}| = r\ddot{\varphi} \quad (44)$$

$$|\mathbf{a}_{\perp}| = r\dot{\varphi}^2 \quad (45)$$