Multidimensional Spaces

In this chapter, we introduce multidimensional spaces, laying the foundation for the core themes of Analysis II.

The Euclidean Structure

In this section, we discuss spaces endowed with a Euclidean structure. We begin by defining the standard vector space $\mathbb{R}^n$ as the set of ordered n-tuples of real numbers:

$$\mathbb{R}^n = \{x = (x_1, \dots, x_n) \mid x_i \in \mathbb{R}\} \tag{1}$$

where $n \in \mathbb{N}$ represents the dimension. The space $\mathbb{R}^n$ is a linear space equipped with the following operations:

- **Vector Addition:** For all $x, y \in \mathbb{R}^n$,

$$x + y = (x_1 + y_1, \dots, x_n + y_n) \tag{2}$$

- **Scalar Multiplication:** For all $\lambda \in \mathbb{R}$ and $x \in \mathbb{R}^n$,

$$\lambda x = (\lambda x_1, \dots, \lambda x_n) \tag{3}$$

Inner Product, Norm, and Distance

The geometric structure of $\mathbb{R}^n$ arises from the introduction of the standard **scalar product** (or dot product), defined as:

$$x \cdot y = \langle x, y \rangle := \sum_{i=1}^{n} x_i y_i \tag{4}$$

Based on the scalar product, we induce the **Euclidean norm** (the length of a vector):

$$\|x\| := \sqrt{\langle x, x \rangle} = \sqrt{\sum_{i=1}^{n} x_i^2} \tag{5}$$

Finally, we define the **Euclidean distance** between two points x and y:

$$d(x, y) := \|y - x\| \tag{6}$$

For all $x, y \in \mathbb{R}^n$, the following inequality holds:

$$|\langle x, y \rangle| \leq \|x\| \cdot \|y\| \tag{7}$$

Proof. If either $x = 0$ or $y = 0$, the inequality holds trivially ($0 \leq 0$). Assume without loss of generality that $x, y \neq 0$. We start from the elementary inequality valid for all real numbers $a, b \in \mathbb{R}$:

$$2ab \leq a^2 + b^2 \tag{8}$$

Let $\lambda > 0$ be an arbitrary scalar. By applying the inequality above to each component, we have:

$$\frac{2x_i y_i}{\lambda} = 2(\lambda x_i)\left(\frac{y_i}{\lambda}\right) \leq \lambda^2 x_i^2 + \frac{y_i^2}{\lambda^2} \tag{9}$$

Summing over $i = 1, \ldots, n$:

$$2\sum_{i=1}^{n} x_i y_i \leq \lambda^2 \sum_{i=1}^{n} x_i^2 + \frac{1}{\lambda^2}\sum_{i=1}^{n} y_i^2 = \lambda^2 \|x\|^2 + \frac{1}{\lambda^2}\|y\|^2 \tag{10}$$

To obtain the tightest bound, we choose $\lambda^2 = \frac{\|y\|}{\|x\|}$. Substituting this back:

$$2\langle x, y\rangle \leq \left(\frac{\|y\|}{\|x\|}\right)\|x\|^2 + \left(\frac{\|x\|}{\|y\|}\right)\|y\|^2 = \|y\|\|x\| + \|x\|\|y\| = 2\|x\|\|y\| \tag{11}$$

Dividing by 2 gives $\langle x, y\rangle \leq \|x\|\|y\|$. The same argument applies to $-x$ and y, proving the absolute value. □

For all $x, y \in \mathbb{R}^n$:

$$\|x + y\| \leq \|x\| + \|y\| \tag{12}$$

This implies the metric triangle inequality: $d(x, z) \leq d(x, y) + d(y, z)$.

Proof. We proceed by squaring the norm:

$$\|x + y\|^2 = \langle x + y, x + y\rangle \tag{13}$$

$$= \langle x, x\rangle + 2\langle x, y\rangle + \langle y, y\rangle \tag{14}$$

$$= \|x\|^2 + 2\langle x, y\rangle + \|y\|^2 \tag{15}$$

By the Cauchy-Schwarz inequality, we know that $\langle x, y\rangle \leq \|x\|\|y\|$. Therefore:

$$\|x + y\|^2 \leq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 \tag{16}$$

$$= (\|x\| + \|y\|)^2 \tag{17}$$

Taking the square root of both sides yields the result. □

Metric Space A metric space is defined as a set endowed with a distance function:

$$(X, d) \tag{18}$$

is a metric space where X is a non-empty set and $d : X \times X \to [0, \infty)$ is a distance function.

A metric space must satisfy three axioms:

- $\forall x, y \in X, \quad d(x,y) = 0 \iff x = y$
- $\forall x, y \in X, \quad d(x,y) = d(y,x)$
- $\forall x, y, z \in X, \quad d(x,z) \leq d(x,y) + d(y,z)$

Some examples of metric spaces are:

- $(\mathbb{R}^n, d_{\text{Euclidean}})$
- $(\mathbb{R}^2, d_{\text{NY}})$ where $d_{\text{NY}}(x,y) = |x_1 - y_1| + |x_2 - y_2|$ is called the New York (or Manhattan) distance.
- Note that for a metric space (X, d), if $Y \subset X$, then $(Y, d|_{Y \times Y})$ is also a metric space.
- The space of continuous functions on a real interval $[a,b]$ with $a < b$, defined as $X = \{f : [a,b] \to \mathbb{R} \mid f \text{ is continuous}\}$. We can endow this set with several metrics, for instance:

$$d_\infty(f,g) = \max_{x \in [a,b]} |f(x) - g(x)| \tag{19}$$

and the integral metric:

$$d_2(f,g) = \left(\int_a^b (f(x) - g(x))^2 \, dx \right)^{\frac{1}{2}} \tag{20}$$

We can define sequences in multidimensional spaces similarly to how we have defined sequences in $\mathbb{R}$.

Sequence Let X be a set. We call a sequence in X a map $x : \mathbb{N} \to X$, and write x_n to indicate the n-th element of the sequence.

There are many notations; the most common are: $(x_n)_{n \geq 0}$, $(x_n)_{n \in \mathbb{N}}$, and $(x_n)_{n=0}^{\infty}$.

Convergent Sequences and Limit Let (X, d) be a metric space. We say that a sequence $(x_n)_{n \geq 0}$ has a limit $x \in X$ if and only if $d(x_n, x) \to 0$ as a sequence of real numbers.

Equivalently, $\forall \varepsilon > 0$, $\exists N > 0$ such that $d(x_n, x) < \varepsilon$ for all $n \geq N$.

We will use this notation:

$$\lim_{n \to \infty} x_n = x \quad \text{or} \quad x_n \to x \tag{21}$$

Let (X, d) be a metric space, and $(x_n)_{n \geq 0}$ a sequence in X. Assume $x_n \to x$ and $x_n \to y$ with $x, y \in X$. Then, $x = y$.

Proof. Assume by contradiction that $x \neq y$. Then $d(x,y) > 0$. Let $\varepsilon = \frac{d(x,y)}{3} > 0$.

By the definition of a limit, since $x_n \to x$, we have:

$$\exists N_x > 0 \quad \text{s.t.} \quad d(x_n, x) < \varepsilon \quad \forall n \geq N_x \tag{22}$$

And since $x_n \to y$:

$$\exists N_y > 0 \quad \text{s.t.} \quad d(x_n, y) < \varepsilon \quad \forall n \geq N_y \tag{23}$$

Now take $N = \max(N_x, N_y)$. Then for all $n \geq N$, applying the triangle inequality yields:

$$3\varepsilon = d(x, y) \leq d(x, x_n) + d(x_n, y) = d(x_n, x) + d(x_n, y) < \varepsilon + \varepsilon = 2\varepsilon \tag{24}$$

This implies $3\varepsilon < 2\varepsilon$, which is a contradiction since $\varepsilon > 0$. Therefore, $x = y$. □

Subsequence Let $(x_n)_{n\geq 0}$ be a sequence in X. We define a subsequence as any sequence of the form $(x_{f(k)})_{k\geq 0}$, where $f : \mathbb{N} \to \mathbb{N}$ is a **strictly increasing** function.

Accumulation Point Let (X, d) be a metric space.

- Given a subset $Y \subset X$, we say that $y \in X$ is an accumulation point (or limit point) of Y if and only if there exists a sequence $(y_n)_{n\geq 0} \subset Y \setminus \{y\}$ such that $y_n \to y$.
- Equivalently, given a sequence $(x_n)_{n\geq 0}$ in X, we say that x is an accumulation point of the sequence if and only if there exists a strictly increasing function $f : \mathbb{N} \to \mathbb{N}$ such that the subsequence $x_{f(k)} \to x$.

Let (X, d) be a metric space and $(x_n)_{n\geq 0} \subset X$ a sequence. Then $(x_n)_{n\geq 0}$ converges to some $x \in X \iff \forall$ subsequence $(x_{n_k})_{k\geq 0}$ we have $x_{n_k} \to x$.

Proof. Recall that if $x_n \to x$ and $x_{n_k} \to y$, then $x = y$ (a subsequence limit must match the sequence limit if the sequence converges). We will prove both implications:

- ($\Leftarrow$) Take as a subsequence the sequence itself (where $n_k = k$). Then $(x_{n_k})_{k\geq 0} = (x_n)_{n\geq 0}$, which by assumption implies $x_n \to x$.
- ($\Rightarrow$) Let $f : \mathbb{N} \to \mathbb{N}$ be a strictly increasing function, so $n_k = f(k)$. The goal is to show that $x_{f(k)} \to x$.

  By definition:

  $$x_n \to x \iff \forall \varepsilon > 0, \exists N > 0 \quad \text{s.t.} \quad d(x_n, x) < \varepsilon \quad \forall n \geq N \tag{25}$$

  Since f is strictly increasing, we have $f(k) \geq k$ for all $k \in \mathbb{N}$. Therefore, for all $k \geq N$, it follows that $f(k) \geq N$. This implies:

  $$d(x_{f(k)}, x) < \varepsilon \quad \forall k \geq N \tag{26}$$

  which means $x_{f(k)} \to x$.

□

Cauchy Sequence A sequence $(x_n)_{n\geq 0}$ is a Cauchy sequence if and only if $\forall \varepsilon > 0, \exists N > 0$ such that $d(x_n, x_m) < \varepsilon$ for all $n, m \geq N$.

Complete Metric Space A metric space (X, d) is complete if and only if every Cauchy sequence in X converges to some limit in X.

Let $(x_m)_{m\geq 0} \subset \mathbb{R}^n$. Then $x_m \to x \in \mathbb{R}^n \iff x_{m,i} \to x_i \quad \forall i \in \{1, \ldots, n\}$.

An element of a sequence in a multidimensional space is defined by its components:

$$x_m = (x_{m,1}, \ldots, x_{m,n}) \tag{27}$$

And its limit is:

$$x = (x_1, \ldots, x_n) \tag{28}$$

Proof.

- $(\Rightarrow)$ Assume $x_m \to x$. By definition, $\forall \varepsilon > 0, \exists N > 0$ such that $\|x_m - x\| < \varepsilon$ for all $m \geq N$. For any component i, this implies:

  $$|x_{m,i} - x_i| = \sqrt{|x_{m,i} - x_i|^2} \leq \sqrt{\sum_{j=1}^{n} |x_{m,j} - x_j|^2} = \|x_m - x\| < \varepsilon \tag{29}$$

  Thus, $x_{m,i} \to x_i$.

- $(\Leftarrow)$ Assume $x_{m,i} \to x_i$ for all $i = 1, \ldots, n$. Let $\varepsilon > 0$. For each i, there exists $N_i > 0$ such that $|x_{m,i} - x_i| < \frac{\varepsilon}{\sqrt{n}}$ for all $m \geq N_i$. Let $N = \max(N_1, \ldots, N_n)$. Then for all $m \geq N$, we have:

  $$\|x_m - x\| = \sqrt{\sum_{i=1}^{n} |x_{m,i} - x_i|^2} < \sqrt{\sum_{i=1}^{n} \left(\frac{\varepsilon}{\sqrt{n}}\right)^2} = \sqrt{n\frac{\varepsilon^2}{n}} = \varepsilon \tag{30}$$

  Thus, $x_m \to x$.

□

Completeness of $\mathbb{R}^n$ The metric space $(\mathbb{R}^n, d_{\text{Euclidean}})$ is **complete**.

Proof. Given a Cauchy sequence $(x_m)_{m\geq 0}$ in $\mathbb{R}^n$, we must show that there exists an $x \in \mathbb{R}^n$ such that $x_m \to x$.

Since $(x_m)_{m\geq 0}$ is a Cauchy sequence with respect to the Euclidean norm, each component sequence $(x_{m,i})_{m\geq 0}$ is a Cauchy sequence in $\mathbb{R}$ for every $i \in \{1, \ldots, n\}$.

Since the field of real numbers $\mathbb{R}$ is complete, every Cauchy sequence in $\mathbb{R}$ converges. Therefore, for each i, there exists a real number $x_i \in \mathbb{R}$ such that $x_{m,i} \to x_i$ as $m \to \infty$.

Let $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. By the previous lemma, convergence component by component implies convergence in the Euclidean metric. Thus, $x_m \to x$. This proves that every Cauchy sequence in $\mathbb{R}^n$ converges to a limit in $\mathbb{R}^n$, meaning the space is complete. □