Work and Energy

Work and Power

Mechanical Work

When the point at which the force acts translates along a path, the force then performs **work**.

Work Work is defined as the product of the **parallel** component of the force along the path and the path itself:

$$W = \int_{r_1(\Gamma)}^{r_2(\Gamma)} \mathbf{F} \cdot d\mathbf{s} = \int_{r_1}^{r_2} F_{\parallel}\ ds \tag{1}$$

where $\mathbf{F}$ is the sum of all external forces.

The unit of measurement of work is the **Joule**

If the force acts perpendicular to the path, then it does no work. For example, the centripetal force that acts during circular motion does no work.

Power

We define power as the rate of change of work with respect to time:

Power

$$P = \frac{\mathrm{d}W}{\mathrm{d}t} \tag{2}$$

The unit of measurement for power is the **Watt**:

$$1\frac{J}{s} = 1\mathrm{W} \tag{3}$$

In the case where the point of application of the force translates with a velocity $\mathbf{v}$, the following relation would hold:

$$P = \frac{\mathbf{F} \cdot d\mathbf{s}}{dt} = \mathbf{F} \cdot \mathbf{v} \tag{4}$$

Kinetic Energy

Assuming that the mass remains constant in the process, we can calculate the work W for a material point:

$$W = \int_{r_1}^{r_2} \mathbf{F} \cdot d\mathbf{r} = \int_{r_1}^{r_2} m\frac{\mathrm{d}\mathbf{v}'}{\mathrm{d}t} \cdot d\mathbf{r} = \int_{r_1}^{r_2} m\frac{\mathrm{d}\mathbf{v}'}{\mathrm{d}t} \cdot \mathbf{v}'dt = m\int_{r_1}^{r_2} \mathbf{v}'d\mathbf{v}' \tag{5}$$

$$\implies W = \frac{m}{2}\mathbf{v}_2^2 - \mathbf{v}_1^2 \tag{6}$$

where $\mathbf{v}_i$ is the velocity at point i. We therefore define the **kinetic energy** K as:

Kinetic Energy

$$K := \frac{1}{2}m\mathbf{v}^2 = \frac{\mathbf{p}^2}{2m} \tag{7}$$

For speeds much smaller than that of light, the following relation holds:

$$W = \Delta K = K(\mathbf{r}_2) - K(\mathbf{r}_1) \tag{8}$$

Potential Energy and Conservative Forces

Conservative Forces

Let $\mathbf{F}(\mathbf{r})$ be a force field and $U(\mathbf{r}) = U(x, y, z)$ a continuous and differentiable function such that:

$$\mathbf{F}(\mathbf{r}) = -\nabla U \tag{9}$$

Then $\mathbf{F}(\mathbf{r})$ is a **conservative** force field.

Potential Energy and Potential

Let us calculate the work done by a force field on a point mass that is translated from $\mathbf{r}_1$ to $\mathbf{r}_2$:

$$W = \int_{r_1}^{r_2} \mathbf{F} \cdot d\mathbf{r} = -\int_{r_1}^{r_2} \nabla U \cdot d\mathbf{r} = -\int_{r_1}^{r_2} dU = U(\mathbf{r}_1) - U(\mathbf{r_2}) = -\Delta U \tag{10}$$

The function $U(\mathbf{r})$ is called the **potential**. We define the **potential energy** as the difference in potential between two points. The potential is a **scalar field**, which is why it is very convenient to use it to analyze interactions that depend on position.

Potential Energy The potential energy $U_{2(1)}$ at the point $\mathbf{r}_2$ with respect to the point $\mathbf{r}_1$ is given by:

$$U_{2(1)} := U(\mathbf{r}_2) - U(\mathbf{r}_1) = -\int_{r_1}^{r_2} \mathbf{F} \cdot d\mathbf{r} \tag{11}$$

In a conservative force field, the work required to move a body depends only on the starting point and the endpoint, and **not** on the path.

Positive work done by a force field leads to a decrease in potential energy.

It is possible to show, using Stokes' theorem, that in a conservative force field the work does not depend on the path:

$$\oint \mathbf{F} \cdot d\mathbf{r} = -\oint \nabla U \cdot d\mathbf{r} = -\int_A \underbrace{\mathrm{rot}(\nabla U)}_{=0} \cdot d\mathbf{A} = 0 \tag{12}$$

Pay attention; time-dependent or velocity-dependent force fields are generally non-conservative.

Conservation of Energy

Fundamental Theorem of Mechanical Energy

In a closed system, the following fundamental theorem holds:

The sum of kinetic energy and potential energy remains **constant** at every instant of time.

In fact:

$$W = \frac{1}{2}m\mathbf{v}_2^2 - \frac{1}{2}m\mathbf{v}_1^2 = U(\mathbf{r}_1) - U(\mathbf{r}_2) \tag{13}$$

$$\iff \left\{\frac{1}{2}m\mathbf{v}_1^2 + U(\mathbf{r}_1)\right\} = \left\{\frac{1}{2}m\mathbf{v}_2^2 + U(\mathbf{r}_2)\right\} = E_{mec} \tag{14}$$

The fundamental theorem of mechanical energy works only in the case of conservative forces. For example, when friction is introduced, conservation no longer holds.

In general, one should take into account several forms of energy, such as rest energy mc^2, thermal energy Q, and so on.

In this way, we can obtain a more general law of conservation of energy:

$$E = mc^2 + K + U + Q + ... = Constant \tag{15}$$

From the law of conservation of energy we also immediately understand that it is impossible to create a *Perpetuum Mobile*. There is no machine that can perform work continuously without being supplied with energy from the outside or without altering the system itself. With no trickery will we ever be able to obtain work for free.