Oscillations

Introduction

Oscillations appear everywhere in nature: in astrophysics, mechanics, quantum mechanics, optics, and of course also in music. All these oscillations have one thing in common: they possess a physical quantity that changes periodically as a function of time. We call such a system an **oscillating system**. It is a physical system in equilibrium that is capable of absorbing a certain amount of energy and using it. This energy is not released immediately but exists in the system in two different forms, between which it oscillates for a certain time.

Harmonic Oscillations

Wherever there is a **restoring force** toward a stable position, a harmonic oscillation is produced.

The Restoring Force: Hooke's Law A restoring force has the form:

$$F(x) = -kx \tag{1}$$

This formula is commonly known as **Hooke's law**, and it represents the restoring force of a spring, where k is a constant that depends on the spring and x is the distance from the neutral position.

When we talk about Hooke's law, we are mainly referring to springs and therefore to an elastic force, but there can also be other restoring forces that we can express using the same relation.

The minus sign in the formula indicates that the force is a restoring force, meaning that it opposes the motion.

Let us now try to write Newton's second law in the case of a point mass m on which an elastic force acts:

$$F(x) = -kx = ma \iff -kx = m\ddot{x} \tag{2}$$

We thus obtain the equation of motion for a harmonic oscillator:

Equation of motion for a simple harmonic oscillator

$$\ddot{x} + \omega_0^2 x = 0 \tag{3}$$

where ω_0 is defined as:

$$\omega_0 := \sqrt{\frac{k}{m}} \tag{4}$$

and is called **natural frequency of the harmonic oscillator**

Note that ω_0 is called the natural frequency, but at the level of its definition it is more similar to an angular velocity than to a frequency.

The equation of motion of a simple harmonic oscillator is a second-order homogeneous differential equation; to solve it, we therefore consider the characteristic equation:

$$\lambda^2 + \omega_0^2 = 0 \implies \lambda_{1,2} = \pm i\omega_0 \tag{5}$$

It follows:

$$x(t) = C_1 e^{i\omega_0 t} + C_2 e^{-i\omega_0 t} \tag{6}$$

Since the exponents are imaginary, we can express this solution in terms of sine and cosine:

$$x(t) = Asin(\omega_0 t) + Bcos(\omega_0 t) \tag{7}$$

With $A, B \in \mathbb{R}$. Using the trigonometric identities:

$$x(t) = Acos(\omega_0 t + \delta) \tag{8}$$

The period of oscillation

The **period** T is defined as the time required to complete one full oscillation.

We know that a sinusoidal function completes a full cycle every 2π radians, therefore:

Period of Oscillation

$$T = \frac{2\pi}{\omega} \tag{9}$$

In the same way, we can define the **frequency** as the number of oscillations per second:

Frequency

$$\nu = \frac{1}{T} = \frac{\omega}{2\pi} \tag{10}$$

The unit of measurement of frequency is the Hertz (Hz) $= s^{-1}$.

Amplitude, velocity, and acceleration

Since the sine (or cosine) function always oscillates between -1 and 1, the coefficient A that multiplies the wave function represents the maximum **amplitude** of oscillation.

Amplitude

$$x(t) = Asin(\omega t + \delta) \implies -A \leq x(t) \leq A \tag{11}$$

The amplitude represents the maximum distance from the origin.

By differentiating with respect to time, we obtain the oscillation **velocity**:

Velocity

$$v(t) = \frac{\mathrm{d}x}{\mathrm{d}t} = A\omega cos(\omega t + \delta) \implies -A\omega \leq v(t) \leq A\omega \tag{12}$$

Similarly, we obtain the acceleration:

Acceleration

$$a(t) = \frac{\mathrm{d}v}{\mathrm{d}t} = -A\omega^2 sin(\omega t + \delta) \implies -A\omega^2 \leq a(t) \leq A\omega^2 \tag{13}$$

Energy balance of the harmonic oscillator

To calculate the energy of the harmonic oscillator, we consider the differential equation of the harmonic oscillator [eq diff oscillatore armonico], and we use $\omega := \omega_0$:

$$\ddot{x} + \omega^2 x = 0 \mid \ \cdot m\dot{x} \tag{14}$$

$$\implies m\dot{x}\frac{\mathrm{d}\dot{x}}{\mathrm{d}t} + m\omega^2 x\frac{\mathrm{d}x}{\mathrm{d}t} = 0 \tag{15}$$

$$\implies m\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\dot{x}^2}{2}\right) + m\omega^2\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{x^2}{2}\right) = 0 \tag{16}$$

$$\implies \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\omega^2 x^2\right) = 0 \tag{17}$$

$$\underbrace{\frac{1}{2}m\dot{x}^2}_{E_{kin}} + \underbrace{\frac{1}{2}m\omega^2 x^2}_{E_{pot}} = cost. := E_{tot} \tag{18}$$

Where:

$$x(t) = Asin(\omega t + \delta) \tag{19}$$

$$\dot{x}(t) = A\omega cos(\omega t + \delta) \tag{20}$$

We then get:

$$E_{tot} = \frac{1}{2}m\omega^2 A^2 cos^2(\omega t + \delta) + \frac{1}{2}m\omega^2 A^2 sin^2(\omega t + \delta) \tag{21}$$

The total energy of the harmonic oscillator is therefore constant with respect to time and is equal to:

Energy of the Harmonic Oscillator

$$E_{tot} = \frac{1}{2} m\omega^2 A^2 \tag{22}$$

It is possible to prove the following relation between the average kinetic and potential energy:

$$\langle E_{kin} \rangle = \langle E_{pot} \rangle = \frac{1}{2} E_{tot} \tag{23}$$

Damped oscillations

The damped harmonic oscillator

Consider a little cart attached to a spring on a horizontal plane; by displacing the cart from its resting position, an elastic force $F = -kx$ will act on it. We then write the equation of motion:

$$m\ddot{x} = -kx \implies \ddot{x} + \frac{k}{m}x = 0 \tag{24}$$

Now we also introduce a frictional force proportional to the velocity of the form $F_a = -\varkappa\dot{x}$. We then obtain:

$$m\ddot{x} = -kx - \varkappa\dot{x} \implies \ddot{x} + \frac{\varkappa}{m}\dot{x} + \frac{k}{m}x = 0 \tag{25}$$

We define the following quantities:

$$\omega_0 = \sqrt{\frac{k}{m}} \quad \text{e} \quad \rho := \frac{\varkappa}{2m} \tag{26}$$

We therefore obtain the equation of the damped harmonic oscillator:

Damped harmonic oscillator equation

$$\ddot{x} + 2\rho\dot{x} + \omega_0^2 x = 0 \tag{27}$$

This is again a second-order linear differential equation. The characteristic polynomial is equal to:

$$\lambda^2 + 2\rho\lambda + \omega_0^2 = 0 \tag{28}$$

The discriminant of this equation is given by:

$$\Delta = 4(\rho^2 - \omega_0^2) \tag{29}$$

We distinguish the following three cases:

1. Weak damping ($\rho < \omega_0$) In this case $\Delta < 0$, therefore the result of the differential equation is:

$$x(t) = (C_1 cos(\omega t) + C_2 sin(\omega t))e^{-\rho t} \tag{30}$$

With both real coefficients $C_{1,2}$ and $\omega = \sqrt{\omega_0^2 - \rho^2}$

The frequency of the damped harmonic oscillator is lower than the frequency of the undamped oscillation.

The equation of the damped harmonic oscillator can also be written in the form:

$$x(t) = Ae^{-\rho t} sin(\omega t + \delta) \tag{31}$$

2. Critical damping ($\rho = \omega_0$)

With $\Delta = 0$, the solution of the differential equation yields:

$$x(t) = (C_1 + C_2 t)e^{-\rho t} \tag{32}$$

3. Overdamping ($\rho > \omega_0$) In this case the discriminant $\Delta > 0$. The result of the differential equation is therefore:

$$x(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} \tag{33}$$

With both real coefficients $\lambda_{1,2} = -\rho \pm \sqrt{\rho^2 - \omega_0^2} < 0$

In all three cases we have:

$$\lim_{t \to \infty} x(t) = 0 \tag{34}$$

So a damped harmonic oscillator (without external forces) will inevitably come to a stop sooner or later.

Energy balance of the damped oscillation

To compute the energy, we multiply the differential equation for the damped oscillator by the momentum $m\dot{x}$:

$$\ddot{x} + 2\rho\dot{x} + \omega_0^2 x = 0 \mid \cdot m\dot{x} \tag{35}$$

$$\implies \frac{\mathrm{d}}{\mathrm{d}t}\{\frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\omega_0^2 x^2\} = -2\rho m\dot{x}^2 = F_R \cdot \dot{x} = P_R \tag{36}$$

where F_R is the friction force and P_R the friction power.

Then the change in time of the total energy is equal to the friction power:

$$\frac{\mathrm{d}}{\mathrm{d}t}\{E_{kin} + E_{pot}\} = P_R \tag{37}$$

We consider now the case of *weak damping*, so that we can make the following approximation:

$$\rho << \frac{1}{T} \tag{38}$$

This means that during one period, the amplitude of the oscillation decreases very slightly, and the exponential function can be, during this time, approximated to 1.

then we have the following:

$$\langle E_{kin}|E_{kin}\rangle_T = \frac{1}{2}E_{tot} = \frac{1}{2}m\left\langle \dot{x}^2\middle|\dot{x}^2\right\rangle_T \tag{39}$$

The friction power during one period is equal to

$$P_R = -2\rho m\left\langle \dot{x}^2\middle|\dot{x}^2\right\rangle_T \implies P_R = -2\rho E_{tot} \tag{40}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}E_{tot} = -2\rho E_{tot} \implies E_{tot} = \underbrace{E_{tot}(0)}_{:=E_0}\cdot e^{-2\rho t} \tag{41}$$

This means that the energy decreases with the decay constant $\tau = \frac{1}{2\rho}$:

$$E_{tot}(t) = E_0 e^{-t/\tau} \tag{42}$$

The relative decrease of the energy during one period is often described thanks to the following Q-factor:

$$Q := 2\pi\frac{E_{tot}}{E_{tot}(t) - E_{tot}(t+T)} = 2\pi\frac{E_0 e^{-t/\tau}}{E_0 e^{-t/\tau}(1 - e^{-T/\tau})} \tag{43}$$

Since $\tau >> T$ we can use an approximation with a first order term as follows:

$$(1 - e^{-T/\tau}) \approx 1 - \left(1 - \frac{T}{\tau}\right) = \frac{T}{\tau} \tag{44}$$

So the Q-factor becomes:

$$Q = 2\pi \frac{\tau}{T} \tag{45}$$

Forced Oscillations, Resonance

Balance solution

Up to this point, we have examined only systems that are displaced once from their equilibrium position and subsequently allowed to undergo free oscillations. We now extend this analysis to encompass oscillatory motion driven by an externally applied *periodic* force (or torque).

Let

$$F_{ext}(t) = F_0 \cos(\Omega t) \quad F_0, \Omega \in \mathbb{R} \tag{46}$$

with Ω the angular frequency of the external periodic force. The differential equation of the forced oscillation is

Forced oscillator equation

$$\ddot{x} + 2\rho\dot{x} + \omega_0^2 x = \frac{F_0}{m} \cos(\Omega t) \tag{47}$$

The solution of a second order non-homogeneous differential equation is the sum of the general solution of the homogeneous equation and one particular solution:

$$x(t) = x_h(t) + x_p(t) \tag{48}$$

We can take as a solution for the homogeneous equation the one from the damped oscillator:

$$x_h(t) = A_0 e^{-\rho t} \cos(\omega t + \delta) \tag{49}$$

Note that $\lim_{t\to\infty} x_h(t) = 0$ So for $t >> \frac{1}{\rho}$ we have $x(t) \to x_p(t)$

The particular solution is equal to:

$$x_p(t) = x_0 e^{i\Omega t} \tag{50}$$

Resonance

If we put the particular solution x_p in the motion equation we get:

$$x_0[-\Omega^2 + 2i\rho\Omega + \omega_0^2]e^{i\Omega t} = \frac{F_0}{m}e^{i\Omega t} \tag{51}$$

Now beware that

$$F_{ext}(t) = Re\{F_0 e^{i\Omega t}\} = F_0 \cos(\Omega t) \tag{52}$$

Then the amplitude:

$$x_0 = \frac{F_0/m}{(\omega_0^2 - \Omega^2) + 2i\rho\Omega} \tag{53}$$

Morover:

$$x_0 := |x_0|e^{i\delta_0} \quad \text{and} \quad a_0 = \frac{F_0}{m} \tag{54}$$

Then the magnitude of the amplitude $|x_0|$ and the phase δ_0 of the forced oscillator are equal to:

$$\frac{|x_0|}{a_0} = \frac{1}{\sqrt{(\omega_0^2 - \Omega^2)^2 + 4\rho^2\Omega^2}} := V(\rho, \Omega) \tag{55}$$

$$\delta_0 = \arctan\left\{\frac{-2\rho\Omega}{\omega_0^2 - \Omega^2}\right\} \tag{56}$$

1) $V(\rho, \Omega)$ is called the **Amplitude Response**