

# Special Relativity

## Galilean Theory of Relativity

The relativity principle has been well known since Galileo Galilei's lifetime. Physicists realized that the motion of objects is not absolute, but rather depends on the observer's frame of reference. In this section, we will introduce the theory of relativity from a classical perspective. We shall then show its limitations and explain the great intuition of Albert Einstein, which led to one of the most successful physical theories of all time.

When discussing classical mechanics, we can state the following postulates:

Postulates of Classical Mechanics

1. **Space and Time:** Space and time are absolute (i.e. homogeneous and isotropic).<sup>1</sup>
2. **Inertial Reference Frames:** A reference frame is called inertial if Newton's laws of motion hold valid within it in their simplest form. If this definition seems too abstract, one can think in terms of accelerated and non-accelerated reference frames. In accelerated reference frames, apparent (fictitious) forces arise, meaning the standard Newton's laws do not hold; hence, they are not inertial. Non-accelerated reference frames, on the other hand, are inertial.
3. **Principle of Relativity:** All inertial reference frames are equivalent for the description of physical laws, i.e., there is no preferred or absolute inertial reference frame.

### Example of Galilean Relativity

To better understand this concept, it is helpful to consider a concrete example. Consider a man travelling on a train moving at a constant velocity  $\mathbf{v}$  along a straight line. Notice how he cannot tell whether he is moving at a constant velocity or if he is perfectly at rest, since there are no net forces acting on him (assuming no vibrations of any kind are coming from the tracks). Now imagine that another train passes by, travelling in the opposite direction with speed  $\mathbf{v}'$ . From the perspective of an observer standing at the station, the first train moves with velocity  $\mathbf{v}$  in one direction and the second train moves with velocity  $\mathbf{v}'$  in the opposite direction. However, from the point of view of the man on the first train, he considers himself to be at rest, and he observes the other train passing by with a relative speed of  $\mathbf{v} + \mathbf{v}'$ .

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<sup>1</sup>Homogeneous means that every observer locally experiences the same space-time structure, regardless of their position; isotropic means that the space-time structure is identical regardless of the direction of observation.

## Events and Worldlines

To delve deeper into the theory of relativity, we must introduce some new conceptual tools.

Event An **event** in the space-time structure is a physical occurrence that takes place at a well-defined time and spatial location. A continuous sequence of events describing the trajectory of a particle through space-time is called the **worldline** of that particle.

We can characterize an event using four coordinates:  $t, x, y, z$ . By multiplying the time  $t$  by the speed of light  $c$ , we can group all four components into a **four-vector** that shares the dimension of length:

$$x_E = \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} \equiv \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}, \quad x_E \in \mathbb{R}^4 \quad (1)$$

We generally adopt the following index notation:

$$\mathbf{x} \equiv x^i, \quad i = 1, 2, 3 \quad (2)$$

where Latin indices denote 3-vectors (spatial components), and

$$\bar{x} \equiv x^\mu, \quad \mu = 0, 1, 2, 3 \quad (3)$$

where Greek indices denote 4-vectors (space-time components). Here,  $\bar{x}$  refers to the **contravariant** four-vector with components  $x^\mu$ , while  $\underline{x}$  refers to the corresponding **covariant** four-vector with components  $x_\mu$ .

## Galilean Transformations

Let us now consider the motion of a particle observed from two different inertial reference frames,  $\mathbf{K}$  and  $\mathbf{K}'$ , whose origins coincide at time  $t = 0$ , and which move with a relative constant velocity  $\mathbf{v}$ . We define the dimensionless velocity vector  $\boldsymbol{\beta}$  as follows:

$$\boldsymbol{\beta} = \frac{\mathbf{v}}{c} = \begin{pmatrix} \beta^1 \\ \beta^2 \\ \beta^3 \end{pmatrix} \quad (4)$$

The **Galilean transformations** provide the relationship between the coordinates of the two reference frames. Since a core postulate of classical mechanics is that time is absolute, we have:

$$ct' = ct \implies t' = t \quad (5)$$

For the spatial coordinates ( $i = 1, 2, 3$ ), the transformation is:

$$x'^i = x^i - \beta^i ct \iff x^i = x'^i + \beta^i x^0 \quad (6)$$

Taking the time derivative of this position transformation yields the classical addition of velocities theorem.

We can use matrix notation to express the complete space-time transformation:

$$\bar{x}' = \mathbb{M}_G \bar{x} \quad (7)$$

where  $\mathbb{M}_G$  is the Galilean boost matrix:

$$\mathbb{M}_G = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\beta^1 & 1 & 0 & 0 \\ -\beta^2 & 0 & 1 & 0 \\ -\beta^3 & 0 & 0 & 1 \end{pmatrix} \quad (8)$$

Notice how the first row  $(1, 0, 0, 0)$  perfectly encodes the absolute time condition  $t' = t$ .

In Galilean relativity, space is absolute, meaning it behaves as a three-dimensional Euclidean space. For a spatial segment of length  $\ell$ , the following relation holds:

$$\ell' = \Delta \mathbf{r}' = (\mathbf{r}_2 - \beta ct) - (\mathbf{r}_1 - \beta ct) = (\mathbf{r}_2 - \mathbf{r}_1) = \Delta \mathbf{r} = \ell \quad (9)$$

**Length Invariance** In a Euclidean space, physical lengths remain invariant (the same) across all inertial reference frames.

Furthermore, as previously established:

$$c\Delta t' = c\Delta t \implies \Delta t' = \Delta t \quad (10)$$

**Time Interval Invariance** Time intervals are absolute and remain invariant across all inertial reference frames in classical mechanics.

We can conclude that within Galilean relativity, the simultaneity of events is absolute. Note that this mathematical structure implies that space and time are two completely separate and independent entities.

### Symmetry of Dynamics

Having discussed kinematics under Galilean transformations, we now briefly turn to dynamics.

**Invariance of Force** The force vector is invariant under Galilean transformations. Mathematically:

$$\mathbf{F}' = m\mathbf{a}' \quad \text{and} \quad \mathbf{F} = m\mathbf{a} \quad (11)$$

where  $\mathbf{F}' = \mathbf{F}$ .

This is easily proven by taking the second time derivative of the spatial coordinates:

$$a'^i = \frac{d^2 x'^i}{dt'^2} = \frac{d^2 x'^i}{dt^2} = \frac{d^2 x^i}{dt^2} - 0 \implies a'^i = a^i \quad (12)$$

This property is known as the **symmetry of dynamics**. In general terms, if  $\bar{q}(t)$  is a four-vector whose components  $q^\nu(t)$  represent a valid solution to the equations of motion in frame  $\mathbf{K}$ , then the transformed vector

$$\bar{q}'(t) = \mathbb{M}_G \bar{q}(t) \quad (13)$$

is an equally valid solution to the equations of motion in frame  $\mathbf{K}'$ . Profound consequences of this dynamic symmetry include the fundamental conservation laws (e.g., conservation of energy and momentum).